Coupled Vibration Analysis of Blades with Angular Pretwist of Cubic Distribution

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This paper presents a finite element model for the vibration analysis of pretwisted uniform cross-sectional blading. The variation of pretwist along the blade length can be in linear or trigonometric increments. The dynamic stiffness for free vibration of the blade is derived from the strain and kinetic energies using Lagrange's equation. The cubic polynomial approximation of the displacements, in two principal directions, is assumed. This method gives excellent results with the use of only a small number of elements. Good agreement is found with the experimental and theoretical results of other investigators for straight and linearly pretwisted blades. The comparison of theoretical results between the linearly and nonlinearly pretwisted beams shows large deviations when the pretwist angle increases.

Nomenclature

A	= area of cross section
A^T	= transpose of matrix A
A^{-1}	= inverse of matrix A
a_1,a_2,a_3	= constant coefficient of pretwist angle along the elemental length
$i,i+1$	= node number of a typical element
K	= kinetic energy
K_G	= blade stiffness matrix
L	= length of blade
ℓ	= length of beam element
M	= element mass matrix
M_G	= blade mass matrix
n	= number of elements
n_i	= ith element
$\underset{\sim}{u}$	= strain energy
U,V	= displacement vectors in XX and YY directions, respectively
u,v	= displacement vectors in xx and yy directions, respectively
X,Y,Z	= principal frame coordinates
x,y,z	= fixed-frame coordinates
α_X,α_Y	= principal flexural rigidities about XX and YY, respectively
α_x,α_y	= flexural rigidity about xx and yy, respectively
α_{xy}	= product flexural rigidity about xx and yy
θ	= angle between principal and fixed frames
λ	= nondimensional frequency parameter, $\omega^2\rho AL^4/\alpha_X$
ρ	= mass per unit volume
ϕ_T	= pretwist angle of the blade
ϕ_{i+1}	= pretwist angle at $i+1$ node
ϕ_i	= pretwist angle at ith node
$\phi_{\mathrm{in}_1},\phi_{\mathrm{in}_2}$	= pretwist angles at the first and second internal nodes of ith element, respectively
$\underset{\sim}{\psi}$	= element nodal displacement vector
$\underset{\approx}{\psi}$	= blade nodal displacement vector
$\underset{\approx}{\psi}', \underset{\approx}{\psi}''$	= differentiation with respect to z
$\dot{\underset{\approx}{\psi}}\ \ddot{\underset{\approx}{\psi}}$	= differentiation with respect to time
ω	= circular frequency

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Introduction

VIBRATION-induced fatigue failure of rotor blades is a problem of major concern to designers of turbomachines. The critical nature of the problem becomes apparent when one notes that there can be as many as a thousand blades of different characteristics in one engine. Therefore, the determination of dynamic characteristics of blading becomes important.

Generally, turbomachinery blades are idealized as tapered and pretwisted beams. In practice, they have asymmetric (airfoil) cross sections and in many cases are so short that they behave more like a curved shell or plate.

Several investigators in the field of turbine blade vibration have studied the vibration characteristics of straight and pretwisted blades under rotating and nonrotating conditions using various methods. All of the investigators who have worked on the vibration analysis of pretwisted blades have considered the pretwist to be linear along the blade length. Carnegie[1] developed a set of equations defining the dynamic motion of pretwisted airfoil blading and investigated the effect of pretwist on the frequencies of vibration using the Rayleigh-Ritz method. Carnegie and Dawson[2,3] further studied the vibration characteristics of airfoil blades by transforming the equations of motion to a set of simultaneous first-order differential equations and then solving them by the Runge-Kutta procedure. Thomas et al.[4] applied the finite element method to the vibration analysis of pretwisted nonuniform blading. A dynamic stiffness matrix was derived for a linearly pretwisted and linearly tapered beam element.

Carnegie[5] derived the equation of motion of rotating airfoil cross-sectional blading allowing for pretwist and stagger angle. Slyper[6] formulated the Stodola method of mode iteration and applied it to pretwisted beams of various width-to-depth ratios. Thomas and Sabuncu[7] studied the vibration characteristics of rotating pretwisted asymmetric cross-sectional blades by using the finite element method. None of the investigators so far have worked on the vibration characteristics of nonlinearly pretwisted blades. The purpose of this work is to investigate the nonlinear pretwist effect on the vibration characteristics of turbine blades.

Theoretical Considerations

Since the element with the cubic polynomial approximation of the displacements, in the two principal directions, gives good results with the use of only a small number of elements, the beam was divided into six elements.

The variation in the pretwist angle along each elemental length is represented by

$$\phi_{i+1}=\phi_i+a_1z+a_2z^2+a_3z^3 \tag{1}$$

The constant coefficients of Eq. (1) (a_1,a_2,a_3) can be calculated if the pretwist angles at four nodes along the elemental length are known. Therefore the element is divided into three subelements. Figure 1 shows a typical beam element. For a linearly pretwisted beam element, angles at each end are calculated as

$$\phi_i=(\phi_T/n)(n_i-1)$$
$$\phi_{i+1}=\phi_i+(\phi_T/n) \tag{2}$$

The pretwist angles at the two internal nodes are calculated using the formulation

$$\phi_{\text{in}_1}=[(\phi_{i-1}-\phi_i)/3]+\phi_i$$
$$\phi_{\text{in}_2}=\tfrac{2}{3}(\phi_{i+1}-\phi_i)+\phi_i \tag{3}$$

For a blade with a trigonometric increment of pretwist angle along the length, the values of pretwist angle at the end nodes and internal nodes are calculated as

$$\phi_i=\phi_T\sin[\pi\ell(n_i-1)/2L]$$
$$\phi_{i-1}=\phi_T\sin(\pi\ell n_i/2L) \tag{4}$$

$$\phi_{\text{in}_1}=\phi_T\sin[\pi\ell(n_i-\tfrac{2}{3})/2L]$$
$$\phi_{\text{in}_2}=\phi_T\sin[\pi\ell(n_i-\tfrac{1}{3})/2L] \tag{5}$$

Energy Expressions

The expressions for strain energy and kinetic energy of a pretwisted beam element are given by Carnegie.[1] Assuming cubic variation of pretwist, these expressions can be written in the *oxy* coordinate system as follows.

Strain Energy

$$\underset{\sim}{U}=\frac{1}{2}\int_0^\ell\left[\alpha_x\left(\frac{\partial^2 v}{\partial z^2}\right)^2+2\alpha_{xy}\left(\frac{\partial^2 v}{\partial z^2}\right)\left(\frac{\partial^2 u}{\partial z^2}\right)+\alpha_y\left(\frac{\partial^2 u}{\partial z^2}\right)^2\right]\mathrm{d}z \tag{6}$$

where

$$\alpha_x=\alpha_X\cos^2\theta+\alpha_Y\sin^2\theta$$
$$\alpha_y=\alpha_Y\cos^2\theta+\alpha_X\sin^2\theta$$
$$\alpha_{xy}=(\alpha_Y-\alpha_X)\sin\theta\cos\theta \tag{7}$$

and

$$\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} \tag{8}$$

or

$$\underset{\sim}{\sigma}=\underset{\sim}{T}\,\underset{\sim}{\Sigma} \tag{9}$$

Using Eq. (8), Eq. (6) can also be written in matrix form as

$$\underset{\sim}{U}=\frac{1}{2}\int_0^\ell\begin{bmatrix} u'' \\ v'' \end{bmatrix}^T\begin{bmatrix} \alpha_y & \alpha_{xy} \\ \alpha_{xy} & \alpha_x \end{bmatrix}\begin{bmatrix} u'' \\ v'' \end{bmatrix}\mathrm{d}z \tag{10}$$

or

$$\underset{\sim}{U}=\frac{1}{2}\int_0^\ell(\underset{\sim}{\sigma}'')^T\underset{\sim}{r}(\underset{\sim}{\sigma}'')\mathrm{d}z \tag{11}$$

where

$$\underset{\sim}{r}=\underset{\sim}{T}\,\underset{\sim}{R}\,\underset{\sim}{T}^T \tag{12}$$

$$R=\begin{bmatrix} \alpha_Y & 0 \\ 0 & \alpha_X \end{bmatrix} \tag{13}$$

The distribution of the displacements U and V over this element is assumed to be given by the cubic polynomials

$$U=Ga_x,\quad V=Ga_y \tag{14}$$

where

$$G=[1,z,z^2,z^3]$$
$$a_x=[a_{x1},a_{x2},a_{x3},a_{x4}]$$
$$a_y=[a_y\;,a_{y2},a_{y3},a_{y4}] \tag{15}$$

These displacement curves contain eight undetermined parameters; hence,

$$\underset{\sim}{\Sigma}=\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}[a_x\quad a_y]^T \tag{16}$$

The nodal displacements of the element $(i,\ i+1)$ are represented by the vector

$$\underset{\sim}{\Psi}=[U_1\ U_1'U_{i+1}\ U_{i+1}'\ V_i\ V_i'\ V_{i+1}\ V_{i+1}']^T$$

By making use of the local geometry of the element, we can write

$$\underset{\sim}{\Psi}=\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}\{a_x\quad a_y\} \tag{17}$$

or

$$\{a_x\quad a_y\}=\begin{bmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}\underset{\sim}{\Psi} \tag{18}$$

where C^{-1} is given in the Appendix. From Eqs. (16) and (18)

$$\underset{\sim}{\Sigma}=\begin{bmatrix} GC^{-1} & 0 \\ 0 & GC^{-1} \end{bmatrix}\underset{\sim}{\Psi} \tag{19}$$

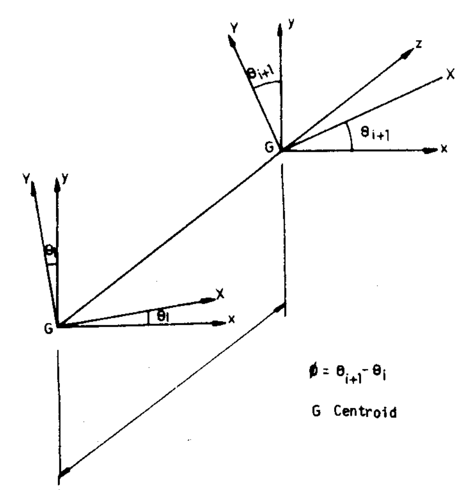

Fig. 1a Typical beam element.

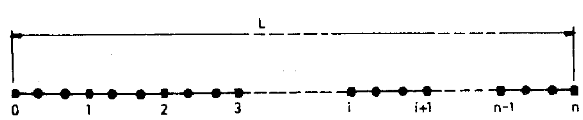

Fig. 1b Segmentation of a beam into elements and subelements (■ elemental nodes, ● subelemental nodes).

Substituting Eq. (19) in the strain energy equation (11) yields

$$\underset{\sim}{U} = (\alpha_X/2)\, \underset{\sim}{\Psi}^T K_E \underset{\sim}{\Psi} \tag{20}$$

where

$$K_E = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

Using the Appendix, submatrices K_{11}, K_{12}, and K_{22} can be written in nondimensional form as

$$K_{11} = \ell^3 [(\alpha_Y/\alpha_X)(A_0 + \delta_1 B_0 + \delta_2 B_1 + \delta_3 B_2 + \delta_4 B_3 + \delta_5 B_4 + \delta_7 C_1 + \delta_8 C_2 + \delta_9 C_3 + \delta_{10} C_4 + \delta_{12} C_6 + \delta_{13} C_7 + \delta_{14} C_8) + \delta_{15} C_0 + \delta_{16} C_1 + \delta_{17} C_2 + \delta_{18} D_0 + \delta_{19} D_1 + \delta_{20} D_2 + 2\delta_1 E_0 + 2\delta_2 E_1 + 8\delta_3 E_2 + \delta_4 E_3 + \delta_5 E_4 + \delta_{11} C_5 (\alpha_Y/\alpha_X)]$$

$$K_{12} = \ell^3 [\delta_{22} F_0 + \delta_{23} F_1 + \delta_{24} C_0 + \delta_{25} C_1 + \delta_{26} C_2 + \delta_{27} C_3 + \delta_{28} C_4 + \delta_{29} C_5 + \delta_{30} G_0 + \delta_{31} G_1 + \delta_{32} G_2 + \delta_{33} H_0 + \delta_{34} H_1 + \delta_{35} H_2 + \delta_{36} H_3 + \delta_{37} H_4 + \delta_{38} H_5 + \delta_{39} H_6 - (\alpha_Y/\alpha_X)(\delta_{22} F_0^T + \delta_{23} F_1^T + \delta_{24} C_0^T + \delta_{25} C_1^T + \delta_{26} C_2^T + \delta_{27} C_3^T + \delta_{28} C_4^T + \delta_{29} C_5^T + \delta_{30} G_0^T + \delta_{31} G_1^T + \delta_{32} G_2^T + \delta_{33} H_0^T + \delta_{34} H_1^T + \delta_{35} H_2^T + \delta_{36} H_3^T + \delta_{37} H_4^T + \delta_{38} H_5^T + \delta_{39} H_6^T)]$$

$$K_{22} = \ell^3 [A_0 + \delta_1 B_0 + \delta_2 B_1 + \delta_3 B_2 + \delta_4 B_3 + \delta_5 B_4 + \delta_6 C_0 + \delta_7 C_1 + \delta_8 C_2 + \delta_9 C_3 + \delta_{10} C_4 + \delta_{11} C_5 + \delta_{12} C_6 + \delta_{13} C_7 + \delta_{14} C_8 + (\alpha_Y/\alpha_X)(\delta_{15} C_0 + \delta_{16} C_1 + \delta_{17} C_2 + \delta_{18} D_0 + \delta_{19} D_1 + \delta_{20} D_2 + \delta_{21} D_3 + 2\delta_1 E_0 + 8\delta_3 E_2 + \delta_4 E_3 + \delta_5 E_4 + 2\delta_2 E_1)]$$

Kinetic Energy

The kinetic energy expression is given by Carnegie[1] as

$$K = \frac{\rho}{2} \int_0^\ell A \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right] dz \tag{21}$$

Using a complete polynomial displacement field that, in the general case, is given in Eq. (14), substituting Eqs. (16-18) in Eq. (14), and resulting in Eq. (21), yields

$$K = \frac{\rho A}{2} \int_0^\ell \dot{\underset{\sim}{\Psi}}^T M \dot{\underset{\sim}{\Psi}} dz \tag{22}$$

where

$$M = \begin{bmatrix} (GC^{-1})^T & 0 \\ 0 & (GC^{-1})^T \end{bmatrix} \begin{bmatrix} GC^{-1} & 0 \\ 0 & GC^{-1} \end{bmatrix}$$

By making use of the Appendix, Eq. (22) can be written in nondimensional form as

$$K = (\rho A \ell/2) 420 \dot{\underset{\sim}{\Psi}}^T M \dot{\underset{\sim}{\Psi}} \tag{23}$$

where

$$M = \begin{bmatrix} M_0 & 0 \\ 0 & M_0 \end{bmatrix}$$

M_0 is given in the Appendix.

The global mass and stiffness matrices are obtained by assembling the element mass and element stiffness matrices individually. Noting that $\underset{\sim}{U}$ is a function of $\underset{\sim}{\Psi}$ only and K is a function of $\dot{\underset{\approx}{\Psi}}$ only, from Lagrange's equation

$$\frac{d\underset{\sim}{U}}{d\underset{\approx}{\Psi}} - \frac{d}{dt}\left(\frac{dK}{d\dot{\underset{\approx}{\Psi}}} \right) = Q$$

This gives rise to the eigenvalue problem

$\{ [K_G] - \lambda [M_G] \} \underset{\approx}{\Psi} = 0$ for the free vibration of the beam

where K_G and M_G are the assembled stiffness matrix and mass matrix of the beam, respectively, and λ the nondimensional frequency parameter.

Discussion of Results

The finite element model developed in this paper is applied to the vibration analysis of uniform cross-sectional blades with and without pretwist in the first instance to compare the results obtained with those of other investigators in order to establish the accuracy of the model. Second, the deviation of the frequencies between linearly and nonlinearly pretwisted similar blades is theoretically shown.

As shown in Table 1, good agreement between the theoretical and experimental results obtained by Carnegie and Dawson,[2] the finite element results of Thomas and Sabuncu,[7] and the author for an airfoil cross-sectional blade. In Table 2, comparison is made of the theoretical and ex-

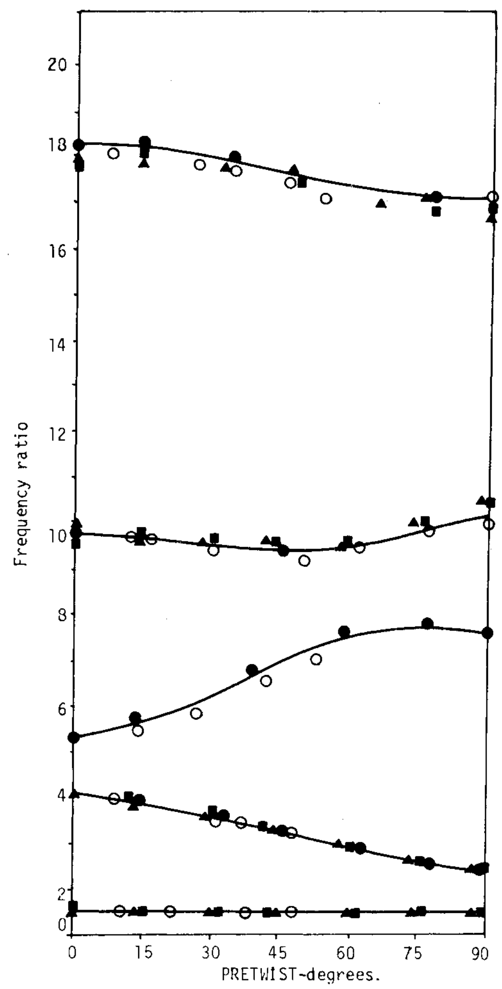

Fig. 2 Lateral vibrations of pretwisted cantilever beams. Experimental: Carnegie ■ CW, ▲ ACW (airfoil section); ○ Bristol Siddeley (rectangular). Calculated: ● Slyper, —— Sabuncu.

Table 1 Comparison of results for a straight uniform blade[a]

Ref. 2			Ref. 7	Sabuncu
Natural frequencies by analytical solution, cycles/s	Natural frequencies by transformation solution, cycles/s	Experimental results, cycles/s	FEM (8 elements), cycles/s	FEM (6 elements), cycles/s
96.9	96.9	97.0	96.9	96.9
607.0	606.5	610.0	607.6	607.6
869.0	868.0	730.0	866.4	866.3
1072.9	1074.2	1102.0	1064.9	—
1699.0	1698.0	1693.0	1702.2	1703.9

[a] $A=0.5896$ cm^2, $I_{XX}=3.49\times10^{-3}$ cm^4, $I_{YY}=2.79\times10^{-1}$ cm^4, $L=15.24$ cm, $E=21.795\times10^5$ kg/cm^2, $\rho=0.0078$ kg/cm^3.

Table 2 Frequency ratios of straight uniform cantilever[a]

	Frequency ratios				Errors, %		
	Slyper			Sabuncu	Based on experiment		
Mode No.	Theoretical Standard	Theoretical Stodola	Experimental	FEM	Standard	Stodola	FEM
1	1.00	1.00	1.00	1.00	—	—	—
2	6.28	6.29	6.09	6.26	3.0	3.3	2.7
3	17.57	17.74	17.11	17.57	2.6	3.7	2.6
4	34.38	35.16	33.59	34.60	2.4	4.7	3.0

[a] $I_Y/I_X=64$, $L=35.56$ cm.

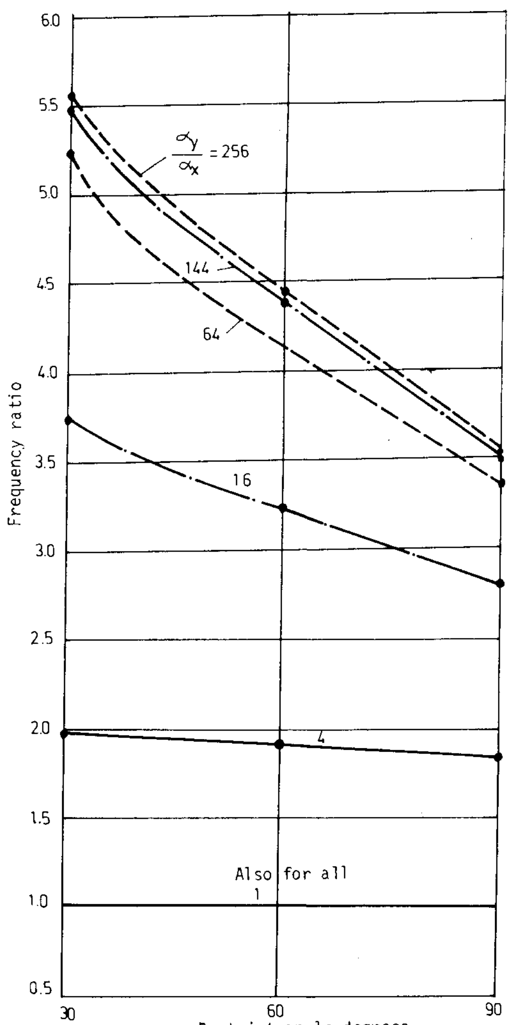

Fig. 3 Vibration of second mode frequency ratios of pretwisted beams: —— Ref. 4, —·— present results ($N=6$).

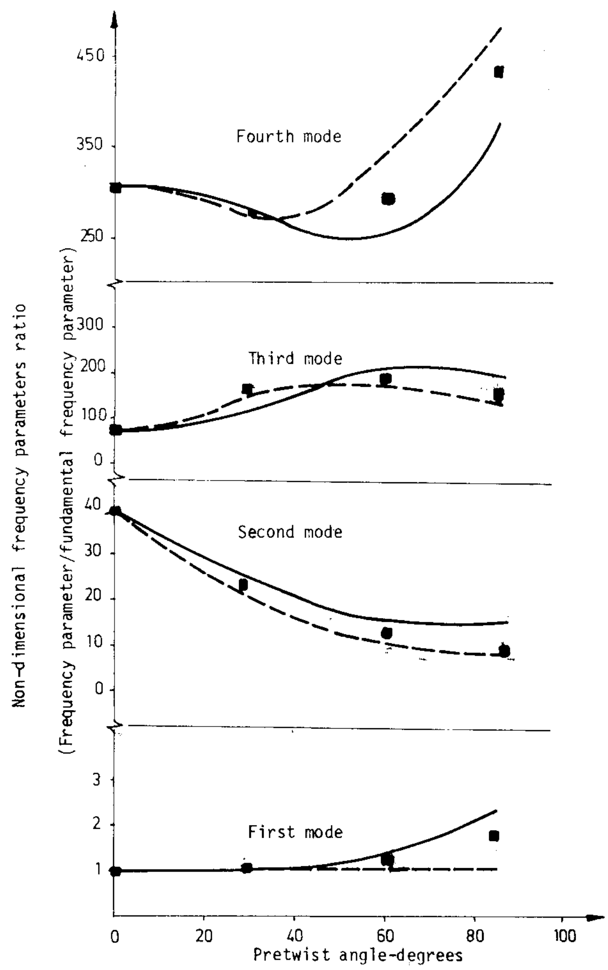

Fig. 4 Frequency ratio against different pretwist angles ($I_Y/I_X=75$, $L=10$ cm): --- linear pretwist, —— nonlinear pretwist, ■ nonlinear pretwist distributed linearly between nodes.

Table 3 Comparison of frequency ratios of pretwisted beams[a]

	1st mode	2nd mode	3rd mode
Ref. 4	1.0438	3.5920	13.8436
Present element ($N=6$)	1.0403	3.6649	14.3429

[a] $\Phi_T = 90$ deg, $I_Y/I_X = 256$, $L = 25.4$ cm.

perimental results obtained by Slyper[6] and the author for a uniform cantilever. Excellent agreement is obtained between the standard mathematical solution and the author's results. In Table 3 comparison of the theoretical results obtained for a 90 deg pretwisted beam by Thomas et al.[4] and the author is also made and good agreement is observed. Figure 2 (from Ref. 6) has been drawn using Carnegie's results for blading of an airfoil cross section for both clockwise (CW) and anticlockwise (ACW) twist. The following particulars apply to these blades: $I_{YY}/I_{XX} = 67.1$, length $L = 15.24$ cm, area $A = 0.5896$ cm^2, and corresponding equivalent width-to-thickness ratio $= 8.19$.

A set of test results (produced by Bristol Siddeley Engines Ltd.) for a beam of uniform rectangular cross section having a width-to-thickness ratio of 8 and the theoretical results obtained by Slyper[6] has also been included. Agreement with the present results is again generally good for the first five critical frequencies investigated. The first endwise mode has been omitted by Carnegie, presumably because it was not readily excited by a chordwise excitation as indicated by Slyper.[6]

Figure 3 compares the second mode frequency ratios calculated by the author and Thomas et al.[4] for various rectangular cross-sectional blades having various linear twists. As can be seen, the agreement between the two theoretical results is good.

Figure 4 compares the nondimensional frequency parameters of a linearly pretwisted blade and a similar blade with a pretwist of trigonometric increment. The trigonometric increment of pretwist angle along the blade length is calculated as $\phi = \phi_T \sin(\pi z/2L)$, where z is the distance measured from the root of a blade. The vibration of pretwisted blades, by means of beam theory, may be analyzed by using as few as four elements with good accuracy. The object of this work is to show the effect of nonlinearity in the pretwist on the frequencies of vibration. To do this, the nondimensional frequency parameter ratios of blades, having the same physical properties and with linear and nonlinear pretwist, are compared for the same number of elements. To show the effect of cubic and linear distribution of pretwist between the elemental nodes on the frequencies of vibration, for the same blade with a trigonometric increment of pretwist, the results of linear distribution of twist, using eight elements, have also been included. As seen from the figure, the rather rough representation of continuity with linear distribution causes a difference in the frequencies of vibration.

The effect of nonlinearity on the frequencies of vibration of the blade increases as the pretwist angle increases.

Conclusions

The finite element model developed in this paper is found to be ideal for the vibration analysis of uniform cross-sectional blades having any form of pretwist. The results obtained by using a very small number of elements give good accuracy when compared with the experimental and theoretical results of other investigators.

Appendix

Variation of pretwist along the beam element is assumed to be

$$\phi_{i+1} = \phi_i + a_1 z + a_2 z^2 + a_3 z^3$$

where a_1, a_2, and a_3 are the constant coefficients of pretwist along each blade element. By making $Z = \ell\eta$ transformation, all of the matrices used in Eqs. (20) and (22) are given in nondimensional form as $z=0$, $\eta=0$; $z=\ell$, $\eta=1$, Hence

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & 1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \quad A_0 = \begin{bmatrix} 12 & 6 & 12 & 6 \\ & 4 & -6 & 2 \\ \text{symmetric} & & 12 & -6 \\ & & & 4 \end{bmatrix} \quad B_0 = \begin{bmatrix} -36 & -18 & 36 & -3 \\ & -4 & 3 & 1 \\ \text{symmetric} & & -36 & 18 \\ & & & -4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -6 & -6 & 36 & -4 \\ & -2 & 6 & 1 \\ \text{symmetric} & & -66 & 30 \\ & & & -6 \end{bmatrix} \quad B_2 = \begin{bmatrix} 12 & -8 & 198 & -1 \\ & -4 & 43 & 3 \\ \text{symmetric} & & -408 & 176 \\ & & & -32 \end{bmatrix} \quad B_3 = \begin{bmatrix} 18 & 0 & 171 & -3 \\ & -1 & 42 & 1 \\ & & -360 & 150 \\ \text{symmetric} & & & -25 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 8 & 1 & 76 & -2 \\ & 0 & 20 & 0 \\ \text{symmetric} & & -160 & 65 \\ & & & -10 \end{bmatrix} \quad C_0 = \begin{bmatrix} 936 & 132 & 324 & -78 \\ & 24 & 78 & -18 \\ \text{symmetric} & & 936 & -132 \\ & & & 24 \end{bmatrix} \quad C_1 = \begin{bmatrix} 144 & 28 & 108 & -24 \\ & 6 & 28 & -6 \\ \text{symmetric} & & 408 & -60 \\ & & & 10 \end{bmatrix}$$

$$C_2=\begin{bmatrix} 76 & 17 & 92 & -19 \\ & 4 & 25 & -5 \\ \text{symmetric} & & 580 & -65 \\ & & & 10 \end{bmatrix} \quad C_3=\begin{bmatrix} 33 & 8 & 57 & -11 \\ & 2 & 16 & -3 \\ \text{symmetric} & & 483 & -49 \\ & & & 7 \end{bmatrix} \quad C_4=\begin{bmatrix} 540 & 138 & 1242 & -225 \\ & 36 & 357 & -63 \\ \text{symmetric} & & 13608 & 1260 \\ & & & 68 \end{bmatrix}$$

$$C_5=\begin{bmatrix} 294 & 78 & 861 & -147 \\ & 21 & 252 & -42 \\ \text{symmetric} & & 11844 & 1008 \\ & & & 126 \end{bmatrix} \quad C_6=\begin{bmatrix} 4464 & 1218 & 16128 & -2604 \\ & 336 & 4788 & -756 \\ \text{symmetric} & & 272160 & -21420 \\ & & & 2520 \end{bmatrix} \quad C_7=\begin{bmatrix} 1377 & 384 & 5994 & -198 \\ & 108 & 1800 & -270 \\ \text{symmetric} & & 121770 & 8910 \\ & & & 990 \end{bmatrix}$$

$$C_8=\begin{bmatrix} 1332 & 378 & 6858 & -999 \\ & 108 & 2079 & -297 \\ \text{symmetric} & & 165132 & -11286 \\ & & & 1188 \end{bmatrix} \quad E_0=\begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ \text{symmetric} & & 36 & -3 \\ & & & 4 \end{bmatrix} \quad E_1=\begin{bmatrix} 36 & 6 & 36 & 0 \\ & 2 & -6 & -1 \\ \text{symmetric} & & 36 & 0 \\ & & & 6 \end{bmatrix}$$

$$E_2=\begin{bmatrix} 72 & 15 & -72 & -6 \\ & 4 & -15 & -3 \\ \text{symmetric} & & 72 & 6 \\ & & & 18 \end{bmatrix} \quad E_3=\begin{bmatrix} 180 & 42 & -180 & 30 \\ & 11 & -42 & -11 \\ \text{symmetric} & & 180 & 30 \\ & & & 65 \end{bmatrix} \quad F_0=\begin{bmatrix} -36 & -33 & 36 & -3 \\ -3 & -4 & 3 & 1 \\ 36 & 3 & -36 & 33 \\ -3 & 1 & 3 & -4 \end{bmatrix}$$

$$F_1=\begin{bmatrix} -3 & -6 & 3 & 3 \\ 0 & -1 & 0 & 1 \\ 33 & 6 & -33 & 27 \\ -3 & 0 & 3 & -3 \end{bmatrix} \quad G_0=\begin{bmatrix} 0 & 10 & 0 & -10 \\ -10 & -5 & 10 & -5 \\ 0 & -10 & 0 & 10 \\ 10 & 5 & 10 & 5 \end{bmatrix} \quad G_1=\begin{bmatrix} -18 & 6 & 18 & -24 \\ -9 & -2 & 9 & -7 \\ 18 & -6 & 18 & 24 \\ 21 & 8 & -21 & 13 \end{bmatrix}$$

$$G_2=\begin{bmatrix} -18 & 0 & 18 & -18 \\ -6 & -1 & 6 & -5 \\ 18 & 0 & -18 & 18 \\ 18 & 6 & 18 & 12 \end{bmatrix} \quad H_0=\begin{bmatrix} -30 & -6 & -30 & 6 \\ 6 & 0 & -6 & 1 \\ 30 & 6 & 30 & -6 \\ -6 & -1 & 6 & 0 \end{bmatrix} \quad H_1=\begin{bmatrix} -78 & -18 & -132 & 24 \\ -4 & -2 & -31 & 5 \\ 78 & 18 & 132 & -24 \\ 11 & -2 & 46 & -2 \end{bmatrix}$$

$$H_2=\begin{bmatrix} -144 & -36 & -360 & 60 \\ -20 & -6 & -92 & 14 \\ 144 & 36 & 360 & -60 \\ 12 & -2 & 180 & -10 \end{bmatrix} \quad H_3=\begin{bmatrix} -76 & -20 & -260 & 40 \\ -14 & -4 & -70 & 10 \\ 76 & 20 & 260 & -40 \\ -2 & 0 & 170 & -10 \end{bmatrix} \quad H_4=\begin{bmatrix} -792 & -216 & -3528 & 504 \\ -168 & -48 & -984 & 132 \\ 792 & 216 & 3528 & 504 \\ 24 & 12 & 2856 & -168 \end{bmatrix}$$

$$H_5=\begin{bmatrix} -5400 & -1512 & -30240 & 4032 \\ -1248 & -360 & -8652 & 1092 \\ 5400 & 1512 & 30240 & 4032 \\ 468 & 168 & 29232 & -1680 \end{bmatrix} \quad H_6=\begin{bmatrix} 3528 & -1008 & -24192 & 3028 \\ -846 & -252 & -7056 & 840 \\ 3528 & 1008 & 24192 & 3028 \\ 504 & 168 & 27216 & -1512 \end{bmatrix}$$

$$M_0=\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}$$

$$\delta_1=-\frac{1}{15}a_1^2\ell^2,\quad \delta_2=-\frac{2}{15}a_1a_2\ell^2,\quad \delta_3=-\frac{2}{105}\ell^2(a_1a_3+a_2^2),\quad \delta_4=-\frac{2}{35}\ell^2a_2a_3,\quad \delta_5=-\frac{3}{35}a_3^2\ell^2,\quad \delta_6=\frac{1}{2520}a_1^4\ell^4$$

$$\delta_7=\frac{1}{210}a_1^3a_2\ell^4,\quad \delta_8=\frac{\ell^4}{315}(a_1^3a_3+3a_1^2a_2^2),\quad \delta_9=\frac{\ell^4}{315}(7a_1^2a_2a_3+4a_1a_2^3),\quad \delta_{10}=\frac{\ell^4}{41580}(17a_1^2a_3^2+64a_1a_2^2a_3+8a_2^4)$$

$$\delta_{11}=\frac{\ell^4}{3465}(7a_1a_2a_3^2+4a_2^3a_3),\quad \delta_{12}=\frac{\ell^4}{30030}(a_1a_3^2-3a_2^2a_3^2),\quad \delta_{13}=\frac{1}{5005}a_2a_3^2\ell^4,\quad \delta_{14}=\frac{1}{20020}a_3^4\ell^4,\quad \delta_{15}=\frac{\ell^4}{630}a_2^2$$

$$\delta_{16}=\frac{\ell^4}{70}a_2a_3,\quad \delta_{17}=\frac{a_3^2}{70}\ell^4,\quad \delta_{18}=2a_1a_2\ell^3,\quad \delta_{19}=\frac{\ell^3}{105}(3a_1a_3+2a_2^2),\quad \delta_{20}=\frac{3a_3a_2}{35}\ell^3$$

$$\delta_{21}=\frac{3}{70}a_3^2\ell^3,\quad \delta_{22}=\frac{a_2}{15}\ell^2,\quad \delta_{23}=\frac{a_3}{5}\ell^2,\quad \delta_{24}=-\frac{a_1a_2^2}{315}\ell^4,\quad \delta_{25}=-\frac{\ell^4}{840}(3a_1^2a_3+4a_2^2a_1)$$

$$\delta_{26}=-\frac{\ell^4}{315}(a_2^3+4a_1a_2a_3),\quad \delta_{27}=-\frac{\ell^4}{105}(a_1a_3^2+2a_2^2a_3),\quad \delta_{28}=-\frac{\ell^4}{924}a_2a_3^2,\quad \delta_{29}=-\frac{\ell^4}{1540}a_3^3,\quad \delta_{30}=\frac{a_1}{5}\ell,\quad \delta_{31}=\frac{2a_2}{15}\ell$$

$$\delta_{32}=\frac{a_3}{5}\ell,\quad \delta_{33}=-\frac{a_1^3}{30}\ell^3,\quad \delta_{34}=-\frac{a_1^2a_2}{3}\ell^3,\quad \delta_{35}=-\frac{\ell^3}{840}(7a_1^2a_3+12a_2^2a_1),\quad \delta_{36}=-\frac{\ell^3}{105}(a_2^3+4a_1a_2a_3)$$

$$\delta_{37}=-\frac{\ell^3}{5040}(7a_1a_3^2+12a_2^2a_3),\quad \delta_{38}=-\frac{\ell^3}{3080}a_2a_3^2,\quad \delta_{39}=-\frac{\ell^3}{6160}a_3^3$$

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